TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 1, January 1996

A PROBLEM OF FÖLDES AND PURI ON THE WIENER PROCESS

Z. SHI

Abstract. Let W be a real-valued Wiener process starting from 0, and $\tau(t)$ be the right-continuous inverse process of its local time at 0. Földes and Puri [3] raise the problem of studying the almost sure asymptotic behavior of $X(t) = \int_0^{\tau(t)} \mathbb{1}_{\{|W(u)| \le \alpha t\}} du$ as t tends to infinity, i.e. they ask: how long does W stay in a tube before "crossing very much" a given level? In this note, both limsup and liminf laws of the iterated logarithm are provided for X.

1. Introduction

Let W be a real-valued Wiener process with $W(0) = 0$. Define its local time process

$$\ell(t,x) = \frac{d}{dx} \int_0^t \mathbb{1}_{\{W(u)<x\}} du,$$

which is almost surely continuous (up to a modification, which is taken for granted) in (t,x) according to Trotter's theorem ([7]). For a modern account of Wiener local times, see for example Revuz and Yor [6, Chap. VI] from stochastic analysis viewpoint, and Révész [5, Part I] through sample paths. We denote $\ell(t,0)$ by $\ell(t)$ for notational convenience, and write

$$\tau(s) = \inf\{\, t > 0 : \ell(t) > s \,\}, \quad s > 0,$$

the right-continuous inverse process of $\ell(t)$. In their recent paper on Wiener processes, Földes and Puri [3] raise the following problem: let

$$X(t) = \int_0^{\tau(t)} \mathbb{1}_{\{|W(u)| \le \alpha t\}} du, \quad t > 0, \tag{1.1}$$

with a fixed constant $\alpha > 0$, what can be said about the almost sure asymptotics of X?

Our answer to the "limsup" part of the question is a law of the iterated logarithm (LIL).

Theorem 1. *For any $\alpha > 0$,*

$$\limsup_{t\to\infty} \frac{X(t)}{t^2 \log\log t} = \frac{8\alpha^2}{\pi^2} \quad a.s. \tag{1.2}$$

For the "liminf" behavior of X, we have an integral test characterizing its lower functions.

Received by the editors December 7, 1994.
1991 *Mathematics Subject Classification.* Primary 60J65; Secondary 60G17.
Key words and phrases. Wiener process (Brownian motion), law of the iterated logarithm.

Theorem 2. *Let $\alpha > 0$, and let $f > 0$ be a non-decreasing function. Then*

$$\mathbb{P}\Big[\, X(t) < \frac{t^2}{f(t)},\ i.o.\,\Big] = \begin{cases} 0 \\ \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} \begin{cases} < \infty \\ \\ = \infty \end{cases}. \tag{1.3}$$

In particular, the following Chung-type LIL holds:

$$\liminf_{t\to\infty} \frac{\log\log t}{t^2} X(t) = \frac{1}{2} \quad a.s. \tag{1.4}$$

Remarks. (i) In (1.3), the symbol "i.o." means "infinitely often" as t tends to infinity.

(ii) It is somewhat surprising that the lower functions of X do not depend on the value of α. Formally (i.e. informally!) taking $\alpha = \infty$ (thus $X(t) = \tau(t)$) in (1.3) and several lines of elementary calculation yield

$$\mathbb{P}\Big[\, \ell(t) > \big(t\, f(t)\big)^{1/2},\ \text{i.o.}\,\Big] = \begin{cases} 0 \\ \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} \begin{cases} < \infty \\ \\ = \infty \end{cases},$$

for any non-decreasing function $f > 0$. We therefore recover the Erdös-Feller-Kolmogorov-Petrov test for the upper functions of W, since the classical Lévy's identity asserts that $\{\, \ell(t);\, t \ge 0\,\}$ and $\{\, \sup_{0\le s\le t} W(s);\, t \ge 0\,\}$ have the same law.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3.

2. Proof of Theorem 1

Let $X(t)$ be defined as in (1.1). It follows from Brownian scaling that

$$X(t) \overset{(d)}{=} t^2 X(1), \tag{2.1}$$

for any $t > 0$ (the symbol "$\overset{(d)}{=}$" denoting identity in distribution). We are first interested in the Laplace transform of $X(1)$.

Lemma 1. *For any $\theta > 0$,*

$$\mathbb{E} \exp\big(-\frac{\theta^2}{2} X(1)\big) = \exp\Big(-\theta\, \tanh(\alpha\theta)\Big). \tag{2.2}$$

Remark. This has previously been obtained by Földes and Puri [3, p. 536] by solving the corresponding Sturm-Liouville equation. Our proof, based on the Ray-Knight theorem, allows us to make use of a Lévy-type formula, thus avoiding further computation.

Proof of Lemma 1. By occupation times formula for local times (see for example Revuz and Yor [6, Corollary VI.1.6]), $X(1) = \int_{-\alpha}^{\alpha} \ell(\tau(1), x) dx$. The Brownian excursion theory confirms the independence between the processes

$$\{\, \ell(\tau(1), x),\ x \ge 0\,\} \text{ and } \{\, \ell(\tau(1), -x),\ x \ge 0\,\},$$

which, according to the second Ray-Knight theorem (Revuz and Yor [6, Theorem XI.2.3]), behave both like a square Bessel process of dimension 0 (i.e. a linear diffusion process with generator $2x\frac{d^2}{dx^2}$, absorbed at 0), starting from 1. Thus by scaling,

$$\begin{aligned}\mathbb{E}\exp\big(-\frac{\theta^2}{2}X(1)\big) &= \Big[\mathbb{E}\exp\Big(-\frac{\theta^2}{2}\int_0^\alpha R^2(x)dx\Big)\Big]^2 \\ &= \Big[\mathbb{E}\exp\Big(-\frac{(\theta\alpha)^2}{2}\int_0^1 \hat{R}^2(s)ds\Big)\Big]^2,\end{aligned}$$

where R and $\hat{R}$ are Bessel processes of dimension 0, with $R(0) = 1$ and $\hat{R}(0) = 1/\sqrt{\alpha}$. Now Lemma 1 is a particular case (taken for dimension 0) of a formula for square Bessel processes presented in Revuz and Yor [6, Corollary XI.1.8]. □

Remark. For connections between quadratic functionals of Bessel processes and the Sturm-Liouville equation, see for example Pitman and Yor [4].

Although it does not seem trivial to deduce the exact upper tail of $X(1)$ from its Laplace transform (2.2), one can manage to obtain the following useful estimate.

Lemma 2. *For any $\alpha > 0$, we have*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}\Big(X(1) > \lambda\Big) = -\frac{\pi^2}{8\alpha^2}. \tag{2.3}$$

Proof of Lemma 2. We begin with the upper bound. Fix $0 < \varepsilon < 1$. From (2.2) it follows using analytic continuation that

$$\mathbb{E}\exp\big(\frac{\theta^2}{2}X(1)\big) = \exp\Big(\theta\,\mathrm{tg}(\alpha\theta)\Big),$$

for $0 < \theta < \pi/2\alpha$. Consequently, for $\lambda > 0$, we have

$$\begin{aligned}\mathbb{P}\Big(X(1) > \lambda\Big) &\le \exp\Big(-\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2}\lambda\Big)\mathbb{E}\exp\Big(\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2}X(1)\Big) \\ &= \exp\Big(-\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2}\lambda + \frac{\pi(1-\varepsilon)}{2\alpha}\mathrm{tg}\big(\frac{\pi(1-\varepsilon)}{2}\big)\Big),\end{aligned}$$

which yields

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}\Big(X(1) > \lambda\Big) \le -\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2}.$$

The upper bound is proved by sending ε to 0^+. For the lower bound, let

$$\begin{aligned}H_\alpha &= \inf\{\, t > 0 : |W(t)| = \alpha \,\}; \\ \sigma_0 &= \inf\{\, t > H_\alpha : W(t) = 0 \,\}.\end{aligned}$$

Observe that when $t \in (H_\alpha, \sigma_0)$, the Wiener process W stays away from 0, and thus $\ell(t)$ does not increase. Consequently, $\sigma_0 \le \tau(1)$ in case $H_\alpha < \tau(1)$. Define

$\hat{W}(t) = W(t+H_\alpha) - W(H_\alpha)$ which is again a standard Wiener process, independent of $\{ W(u);\, 0 \le u \le H_\alpha \}$. For any x, write $\hat{T}_x = \inf\{ t > 0 : \hat{W}(t) = x \}$. Obviously,

$$\begin{aligned} X(1) &\ge \mathbb{1}_{\{H_\alpha < \tau(1)\}} \int_{H_\alpha}^{\sigma_0} \mathbb{1}_{\{|W(u)| \le \alpha\}} du \\ &= \mathbb{1}_{\{H_\alpha < \tau(1),\, W(H_\alpha) = -\alpha\}} \int_0^{\hat{T}_\alpha} \mathbb{1}_{\{\hat{W}(u) \ge 0\}} du \\ &\qquad + \mathbb{1}_{\{H_\alpha < \tau(1),\, W(H_\alpha) = \alpha\}} \int_0^{\hat{T}_{-\alpha}} \mathbb{1}_{\{\hat{W}(u) \le 0\}} du. \end{aligned}$$

By the strong Markov property and symmetry, we obtain:

$$\begin{aligned} \mathbb{P}\Big(X(1) > \lambda \Big) &\ge \mathbb{P}\Big(H_\alpha < \tau(1) \Big) \mathbb{P}\Big(\int_0^{T_\alpha} \mathbb{1}_{\{W(u) \ge 0\}} du > \lambda \Big) \\ &= \mathbb{P}\Big(H_\alpha < \tau(1) \Big) \mathbb{P}\Big(\int_0^{T_1} \mathbb{1}_{\{W(u) \ge 0\}} du > \frac{\lambda}{\alpha^2} \Big), \end{aligned}$$

with $T_x = \inf\{ t > 0 : W(t) = x \}$. According to the first Ray-Knight theorem (Revuz and Yor [6, Theorem XI.2.2]), $\{ \ell(T_1, 1-x);\, 0 \le x \le 1 \}$ is a two-dimensional square Bessel process (i.e. the square of Euclidean modulus of a two-dimensional Wiener process), starting from 0. Since

$$\int_0^{T_1} \mathbb{1}_{\{W(u) \ge 0\}} du = \int_0^1 \ell(T_1, 1-x) dx,$$

and since a two-dimensional square Bessel process is (stochastically) greater than the square of a real-valued Wiener process, we have

$$\mathbb{P}\Big(X(1) > \lambda \Big) \ge \mathbb{P}\Big(H_\alpha < \tau(1) \Big) \mathbb{P}\Big(\int_0^1 W^2(u) du > \frac{\lambda}{\alpha^2} \Big).$$

Recall the Cameron-Martin formula ([1]):

$$\mathbb{P}\Big(\int_0^1 W^2(u) du > x \Big) \sim \frac{\sqrt{32}}{\pi^2} x^{-1/2} \exp(-\frac{\pi^2}{8} x), \quad x \to \infty,$$

(the symbol "$a(x) \sim b(x)$" ($x \to x_0$) denoting $\lim_{x \to x_0} a(x)/b(x) = 1$). Consequently, for sufficiently large λ,

$$\mathbb{P}\Big(X(1) > \lambda \Big) \ge \frac{C(\alpha)}{\lambda^{1/2}} \exp\Big(-\frac{\pi^2 \lambda}{8\alpha^2} \Big),$$

with $C(\alpha) > 0$ a finite constant depending only on α. This implies the lower bound in Lemma 2. □

Proof of Theorem 1. Let us begin with the upper bound. Fix a small $\varepsilon > 0$. Define $t_n = (1+\varepsilon)^n$. Then $(1+5\varepsilon) t_n^2 / t_{n+1}^2 \ge 1 + 2\varepsilon$. By scaling property (2.1), we have

$$\begin{aligned} &\mathbb{P}\Big(X(t_{n+1}) > (1+5\varepsilon) \frac{8\alpha^2}{\pi^2} t_n^2 \log\log t_n \Big) \\ &\quad = \mathbb{P}\Big(X(1) > (1+5\varepsilon) \frac{8\alpha^2}{\pi^2} \frac{t_n^2}{t_{n+1}^2} \log\log t_n \Big) \\ &\quad \le \mathbb{P}\Big(X(1) > (1+2\varepsilon) \frac{8\alpha^2}{\pi^2} \log\log t_n \Big). \end{aligned}$$

Using (2.3), the above expression is, when n is sufficiently large, bounded above by

$$\exp\Big(-(1+\varepsilon)\log\log t_n\Big) = \frac{1}{n^{1+\varepsilon}\big(\log(1+\varepsilon)\big)^{1+\varepsilon}},$$

which sums for n. It follows from the Borel-Cantelli lemma that (almost surely) for large n, $X(t_{n+1}) \le (1+5\varepsilon)(8\alpha^2/\pi^2)t_n^2 \log\log t_n$. Let $t \in [t_n, t_{n+1}]$. Then

$$\frac{X(t)}{t^2 \log\log t} \le \frac{X(t_{n+1})}{t_n^2 \log\log t_n} \le (1+5\varepsilon)\frac{8\alpha^2}{\pi^2}.$$

Accordingly,

$$\limsup_{t\to\infty} \frac{X(t)}{t^2\log\log t} \le (1+5\varepsilon)\frac{8\alpha^2}{\pi^2},$$

for any $\varepsilon > 0$. Letting ε tend to 0^+ gives the upper bound in Theorem 1. It remains to verify the lower bound part. Fix again an $\varepsilon > 0$. Let $t_n = 2^n$ and let

$$A_n = \Big\{ \int_{\tau(t_{n-1})}^{\tau(t_n)} \mathbb{1}_{\{|W(u)|\le \alpha t_n\}} du > (1-\varepsilon)\frac{8\alpha^2}{\pi^2} t_n^2 \log\log t_n \Big\}.$$

The measurable events (A_n) are obviously independent. Moreover,

$$\mathbb{P}(A_n) = \mathbb{P}\Big(\int_0^{\tau(t_n - t_{n-1})} \mathbb{1}_{\{|W(u)|\le \alpha t_n\}} du > (1-\varepsilon)\frac{8\alpha^2}{\pi^2} t_n^2 \log\log t_n \Big),$$

using the strong Markov property. By scaling, we have

$$\begin{aligned}\mathbb{P}(A_n) &= \mathbb{P}\Big(\int_0^{\tau(1)} \mathbb{1}_{\{|W(u)|\le \alpha t_n/(t_n - t_{n-1})\}} du > (1-\varepsilon)\frac{8\alpha^2}{\pi^2} \frac{t_n^2}{(t_n - t_{n-1})^2} \log\log t_n \Big)\\ &= \mathbb{P}\Big(\int_0^{\tau(1)} \mathbb{1}_{\{|W(u)|\le 2\alpha\}} du > (1-\varepsilon)\frac{32\alpha^2}{\pi^2} \log\log t_n \Big),\end{aligned}$$

using the fact that $t_n/(t_n - t_{n-1}) = 2$. Applying (2.3) to 2α (instead of α) implies that for large n,

$$\mathbb{P}(A_n) \ge \exp\big(-\log\log t_n\big) = \frac{1}{n\log 2},$$

which forms a divergent series. Since the A_n's are independent, the Borel-Cantelli lemma tells $\mathbb{P}(A_n, \text{ i.o.}) = 1$. Therefore

$$\liminf_{t\to\infty} \frac{X(t)}{t^2\log\log t} \ge (1-\varepsilon)\frac{8\alpha^2}{\pi^2} \quad \text{a.s.}$$

Since the constant $\varepsilon > 0$ can be arbitrarily small, this yields the lower bound in Theorem 1. □

3. Proof of Theorem 2

Let us keep the notation introduced previously. The first step is to establish the lower tail of $X(1)$.

Lemma 3. *We have, for any $\alpha > 0$,*

$$\mathbb{P}\Big(X(1) < x \Big) \sim \big(\frac{2}{\pi}\big)^{1/2} x^{1/2} \exp\big(-\frac{1}{2x}\big), \quad x \to 0. \tag{3.1}$$

Consequently, there exists a finite constant $C > 0$ such that

$$\mathbb{P}\Big(X(1) < x \Big) \le C \exp\big(-\frac{1}{2x}\big), \quad \forall x > 0. \tag{3.2}$$

The proof of Lemma 3 is based on a general complex analysis argument I have learnt from Csáki [2, p. 210], which is stated as follows.

Lemma 4. *Let $g(z)$ be holomorphic on $\{ (\mathrm{Im}(z))^2 > 4\varepsilon(\varepsilon - \mathrm{Re}(z)) \}$ for any sufficiently small $\varepsilon > 0$, and let $G(z) = g(z^2)$. Assume that for $t \to \infty$ there exist finite constants $\beta \ge 0$, $\gamma > 1$ and $\eta > 1/2$ such that*

$$G(t+iu) = o\Big(G(t) \exp\big(\beta |u| t^{-1/2}\big)\Big) \quad \textit{uniformly for } |u| \ge t^\eta, \tag{3.3}$$

$$G''(t+iu) = o\big(t^{-\gamma} G(t)\big) \quad \textit{uniformly for } |u| \le t^\eta. \tag{3.4}$$

Then for any $c > 0$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-1/2} g(p) \exp\big(ps - 2p^{1/2}\big) dp = \frac{1 + o\big(s^{(\gamma-1)/2}\big)}{(\pi s)^{1/2}} g\big(s^{-2}\big) \exp\big(-\frac{1}{s}\big), \tag{3.5}$$

as $s \to 0^+$.

Proof of Lemma 3. From (2.2), we have, using integration by parts,

$$\int_0^\infty dx\, e^{-\theta x} \mathbb{P}\Big(X(1) < x \Big) = \frac{1}{\theta} \exp\Big(-(2\theta)^{1/2} \tanh\big(\alpha (2\theta)^{1/2}\big) \Big),$$

for $\theta > 0$. Inverting the Laplace transform yields

$$\begin{aligned} \mathbb{P}\Big(X(1) < x \Big) &= \frac{1}{2\pi i} \int_{2c-i\infty}^{2c+i\infty} \frac{d\theta}{\theta} e^{x\theta} \exp\Big(-(2\theta)^{1/2} \tanh\big(\alpha (2\theta)^{1/2}\big) \Big) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dp}{p} e^{2px} \exp\Big(-2p^{1/2} \tanh\big(2\alpha p^{1/2}\big) \Big) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-1/2} g(p) \exp\big(2px - 2p^{1/2}\big) dp, \end{aligned} \tag{3.6}$$

where

$$g(p) = p^{-1/2} \exp\Big(2p^{1/2} - 2p^{1/2} \tanh\big(2\alpha p^{1/2}\big) \Big).$$

Obviously, $g(z)$ is holomorphic on $\{ (\mathrm{Im}(z))^2 > 4\varepsilon(\varepsilon - \mathrm{Re}(z)) \}$ for any $\varepsilon > 0$. Let

$$G(z) = g(z^2) = \frac{1}{z} \exp\Big(2z - 2z \tanh(2\alpha z) \Big) = \frac{1}{z} \exp\Big(\frac{4z}{e^{4\alpha z} + 1} \Big).$$

We now verify conditions (3.3) and (3.4) for G. Write $G(z) = z^{-1}\exp(H(z))$, with $H(z) \equiv 4z/(e^{4\alpha z}+1)$. Then

$$\begin{aligned}\operatorname{Re}\big(H(t+iu)\big) - H(t) &\le \Big| H(t+iu) - H(t) \Big| \\ &= \frac{|4te^{4\alpha t}(1-e^{4i\alpha u}) + 4iu(e^{4\alpha t}+1)|}{|e^{4\alpha(t+iu)}+1|\,(e^{4\alpha t}+1)} \\ &\le \frac{8t+4|u|}{e^{4\alpha t}-1}. \end{aligned} \tag{3.7}$$

Since $(8t+4|u|)/(e^{4\alpha t}-1) \le |u|t^{-1/2}$ for any $|u| \ge t \ge t_0$ (t_0 depending on the value of α), the above inequality trivially implies (3.3) with $\beta = 1$ and $\eta = 1$. For (3.4), observe that $G''(z) = z^{-2}\big(2 - 2zH'(z) + z^2H''(z) + z^2(H'(z))^2\big)G(z)$. Assume $|u| \le t$ and $t \ge t_0$. By (3.7), we have $\operatorname{Re}\big(H(t+iu)\big) - H(t) \le 1$. Thus $|G(t+iu)| \le 3G(t)$. Several lines of elementary calculation show that

$$\Big| 2 - 2(t+iu)H'(t+iu) + (t+iu)^2H''(t+iu) + (t+iu)^2(H'(t+iu))^2 \Big| \le 3.$$

Consequently, (3.4) holds with $\gamma = 3/2$ (actually any constant strictly smaller than 2 will do). Applying (3.5) to $s = 2x$ and using (3.6) completes the proof of Lemma 3. □

Proof of Theorem 2. The convergent half is an easy consequence of Lemma 3. Indeed, assume f to be non-decreasing such that

$$\int^{\infty} \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} < \infty. \tag{3.8}$$

Thus $f(t) \uparrow \infty$ as $t \uparrow \infty$. Pick a large initial value t_0 and define the sequence (t_n) by recurrence: $t_{n+1} = t_n(1 + 1/f(t_n))$ for $n = 0, 1, 2, \cdots$. Obviously (t_n) increases to infinity. A standard argument shows that (3.8) implies

$$\sum_n f^{-1/2}(t_n) e^{-f(t_n)/2} < \infty. \tag{3.9}$$

By scaling property (2.1) and small deviation probability estimate (3.1), we have

$$\begin{aligned}\mathbb{P}\Big(X(t_n) < \frac{t_n^2}{f(t_n)-3} \Big) &= \mathbb{P}\Big(X(1) < \frac{1}{f(t_n)-3} \Big) \\ &\le \big(f(t_n)-3\big)^{-1/2} \exp\Big(-\frac{f(t_n)-3}{2} \Big) \\ &\le 5 f^{-1/2}(t_n) e^{-f(t_n)/2}, \end{aligned}$$

which, according to (3.9), is summable. Applying the Borel-Cantelli lemma gives that (almost surely) for sufficiently large n, $X(t_n) \ge t_n^2/\big(f(t_n)-3\big)$. Let $t \in [t_n, t_{n+1}]$. Then

$$X(t) \ge X(t_n) \ge \frac{t_n^2}{f(t_n)-3} = \frac{t_{n+1}^2}{\big(1+1/f(t_n)\big)^2\big(f(t_n)-3\big)} \ge \frac{t_{n+1}^2}{f(t_n)} \ge \frac{t^2}{f(t)},$$

implying the convergent half of Theorem 1. To check the divergent half, let $f > 0$ be non-decreasing with

$$\int^{\infty} \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} = \infty. \tag{3.10}$$

In light of (1.4), let us assume without loss of generality that

$$\log\log t \le f(t) \le 3 \log\log t. \tag{3.11}$$

For an elegant argument justifying (3.11), we refer to Csáki [2]. Let $\rho > 0$ and $\varepsilon > 0$. Fix a large initial value $i_0 \equiv i_0(\rho, \varepsilon)$. Define $t_i = \exp(\rho i / \log i)$ (for $i \ge i_0$). By means of (3.11), we have

$$1 + \frac{\rho}{2\log i} \le \frac{t_i}{t_{i-1}} \le 1 + \frac{2\rho}{\log i}, \tag{3.12}$$

$$\frac{1}{2}\log i \le f(t_i) \le 3 \log i, \quad i > i_0. \tag{3.13}$$

Moreover, it is easily seen from (3.10) that $\sum_i f^{-1/2}(t_i) \exp(-f(t_i)/2) = \infty$. Now consider

$$A_i = \Big\{ \frac{t_{i-1}^2}{f(t_i)} \le X(t_i) < \frac{t_i^2}{f(t_i)} \Big\}.$$

Using (2.1), (3.1), (3.12) and (3.13) gives

$$\begin{aligned} \mathbb{P}(A_i) &= \mathbb{P}\Big[\frac{t_{i-1}^2}{t_i^2 f(t_i)} \le X(1) < \frac{1}{f(t_i)} \Big] \\ &\ge (1-\varepsilon)\Big(\frac{2}{\pi}\Big)^{1/2} \Big(f^{-1/2}(t_i) e^{-f(t_i)/2} - \Big(\frac{t_{i-1}^2}{t_i^2 f(t_i)}\Big)^{1/2} \exp\Big(-\frac{t_i^2 f(t_i)}{2t_{i-1}^2}\Big) \Big) \\ &\ge (1-\varepsilon)(1 - e^{-\rho/4})(2/\pi)^{1/2} f^{-1/2}(t_i) e^{-f(t_i)/2}, \end{aligned} \tag{3.14}$$

for any $n > i_0$. Consequently,

$$\sum_i \mathbb{P}(A_i) = \infty. \tag{3.15}$$

Pick $i_0 < i < j$. In view of the strong Markov property, we have

$$\begin{aligned} \mathbb{P}(A_i A_j) &\le \mathbb{P}\Big(A_i, \int_{\tau(t_i)}^{\tau(t_j)} \mathbb{1}_{\{|W(u)| \le \alpha t_j\}} du < \frac{t_j^2}{f(t_j)} - \frac{t_{i-1}^2}{f(t_i)} \Big) \\ &\le \mathbb{P}(A_i)\, \mathbb{P}\Big(\int_0^{\tau(t_j - t_i)} \mathbb{1}_{\{|W(u)| \le \alpha (t_j - t_i)\}} du < \frac{t_j^2 - t_{i-1}^2}{f(t_j)} \Big). \end{aligned}$$

The above estimate together with scaling property (2.1) readily yield

$$\mathbb{P}(A_i A_j) \le \mathbb{P}(A_i)\, \mathbb{P}\Big(X(1) < \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_j)} \Big), \tag{3.16}$$

for any $i_0 < i < j$. Define

$$\mathcal{E}(n) = \Big\{ i_0 < i < j \le n : \, j - i < (\log i)^3 \Big\},$$
$$\mathcal{F}(n) = \Big\{ i_0 < i < j \le n : \, j - i \ge (\log i)^3 \Big\}.$$

Remark that when $i < j < i + (\log i)^3$,

$$\frac{j}{\log j} - \frac{i}{\log i} = \frac{(j-i)\log i - i\log(1+(j-i)/i)}{\log i \, \log j} \sim \frac{j-i}{\log j} \sim \frac{j-i}{\log i},$$

as $i \to \infty$. Let $(i,j) \in \mathcal{E}(n)$. Then by the above observation, we have

$$\exp\Big(-\frac{2\rho(j-i)}{\log i} \Big) \le \frac{t_{i-1}}{t_j} \le \frac{t_i}{t_j} \le \exp\Big(-\frac{\rho(j-i)}{2\log i} \Big).$$

Therefore by (3.13),

$$\begin{aligned} \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_j)} \le \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_i)} &\le \frac{1 - \exp\big(-4\rho(j-i)/\log i\big)}{\big(1 - \exp\big(-\rho(j-i)/2\log i\big)\big)^2} \frac{2}{\log i} \\ &\le \frac{C_1}{\min(j-i, \, \log i)}, \end{aligned}$$

for some finite constant $C_1 > 0$ depending only on ρ and ε (i_0 depending on ε). Using (3.16) and (3.2), we obtain (writing $C_2 \equiv 1/(2C_1)$ in the sequel):

$$\begin{aligned} \mathbb{P}(A_i A_j) &\le C\mathbb{P}(A_i)\exp\big(-C_2 \min(j-i, \, \log i)\big) \\ &\le C\mathbb{P}(A_i)\exp\big(-C_2(j-i)\big) + C\mathbb{P}(A_i)\exp\big(-C_2 \log i\big), \end{aligned}$$

(C being the constant introduced in (3.2)) which in turn implies

$$\begin{aligned} \sum\sum_{(i,j)\in\mathcal{E}(n)} \mathbb{P}(A_i A_j) &\le C \sum_{i=i_0+1}^{n} \mathbb{P}(A_i) \sum_{j>i} e^{-C_2(j-i)} \\ &\quad + C \sum_{i=i_0+1}^{n} \mathbb{P}(A_i) \sum_{i<j<i+(\log i)^3} i^{-C_2} \\ &\le C_3 \sum_{i=i_0+1}^{n} \mathbb{P}(A_i), \end{aligned} \tag{3.17}$$

for some finite constant $C_3 > 0$. Now let $(i,j) \in \mathcal{F}(n)$. In this case, $j - (\log j)^2 \ge i + (\log i)^3 - \big(\log(i + (\log i)^3)\big)^2 \ge i$. Thus $j - i \ge (\log j)^2$. It is noticed that

$$\frac{j}{\log j} - \frac{i}{\log i} = \frac{(j-i)\log i - i\log(1+(j-i)/i)}{\log i \, \log j} \ge \frac{j-i}{2\log j}.$$

Therefore, we have

$$\frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_j)} \le \frac{1}{(1 - t_i/t_j)^2 f(t_j)} \le \frac{1}{\big(1 - \exp(-\rho(j-i)/2\log j)\big)^2 f(t_j)}$$
$$\le \frac{1}{\big(1 - (\log j)^{-2}\big) f(t_j)}.$$

According to (3.16) and (3.1), we get

$$\mathbb{P}(A_i A_j) \le (1+\varepsilon)\mathbb{P}(A_i)\Big(\frac{2}{\pi}\Big)^{1/2} f^{-1/2}(t_j)\exp\Big(-\frac{\big(1-(\log j)^{-2}\big)f(t_j)}{2}\Big).$$

Since $(\log j)^{-2} f(t_j)/2 \le 3/2 \log j \le \varepsilon$ for $j \ge i_0$, the above estimate together with (3.14) confirm

$$\mathbb{P}(A_i A_j) \le \frac{e^{\varepsilon}(1+\varepsilon)}{(1-\varepsilon)(1-e^{-\rho/4})}\mathbb{P}(A_i)\mathbb{P}(A_j),$$

which, with the aid of (3.17) and (3.15), yields

$$\liminf_{n\to\infty} \sum_{i=i_0+1}^{n} \sum_{j=i_0+1}^{n} \mathbb{P}(A_i A_j) \Big/ \Big(\sum_{i=i_0+1}^{n} \mathbb{P}(A_i)\Big)^2 \le \frac{1}{1-e^{-\rho/4}}.$$

Since $\sum_i \mathbb{P}(A_i)$ diverges, using a well-known version of the Borel-Cantelli lemma (see for example Révész [5, p. 28]) gives $\mathbb{P}(A_i, \text{i.o.}) \ge 1 - e^{-\rho/4}$. A fortiori, we have

$$\mathbb{P}\Big[\, X(t) < \frac{t^2}{f(t)},\ \text{i.o.}\,\Big] \ge 1 - e^{-\rho/4},$$

for any $\rho > 0$. The proof of the divergent half is completed by sending ρ to infinity.□

Acknowledgements

I am grateful to Marc Yor for helpful comments on the first draft of this work. Many thanks are due to an anonymous referee for a careful reading of the manuscript and for valuable suggestions.

References

1. R.H. Cameron and W.T. Martin, *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions*, J. Math. Phys. **23** (1944), 195–209. MR **6:**132a
2. E. Csáki, *An integral test for the supremum of Wiener local time*, Probab. Th. Rel. Fields **83** (1989), 207–217. MR **91a:**60206
3. A. Földes and M.L. Puri, *The time spent by the Wiener process in a narrow tube before leaving a wide tube*, Proc. Amer. Math. Soc. **117** (1993), 529–536. MR **93d:**60131
4. J.W. Pitman and M. Yor, *A decomposition of Bessel bridges*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **59** (1982), 425–457. MR **84a:**60091
5. P. Révész, *Random Walk in Random and Non-Random Environments*, World Scientific, Singapore, 1990. MR **92c:**60096
6. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 2nd ed., Springer, Berlin, 1994. CMP 95:04
7. H.F. Trotter, *A property of Brownian motion paths*, Illinois J. Math. **2** (1958), 425–433. MR **20:**2795

Université Paris VI, L.S.T.A. - CNRS URA 1321, Université Paris VI, Tour 45-55, 4 Place Jussieu, F-75252 Paris Cedex 05, France

E-mail address: shi@ccr.jussieu.fr